Polarisation properties of 2ω radiation in plane wave scattering by bound electrons

1987 J. Phys. A: Math. Gen. 20 4191

(http://iopscience.iop.org/0305-4470/20/13/025)

J. Phys. A: Math. Gen. **20** (1987) 4191-4201. Printed in the UK

Polarisation properties of 2ω radiation in plane wave scattering by bound electrons

V P Verma and Vachaspati

Department of Physics, University of Allahabad, Allahabad 211 002, India

Received 20 October 1986, in final form 13 March 1987

Abstract. It is shown that, when plane polarised light of frequency ω and polarisation angle ϕ_0 is scattered by harmonically bound electrons of natural frequency ω_0, the polarisation angle, ϕ, of the second harmonic (2ω) scattered light is given by

$$\tan\phi = \begin{cases} \tan\phi_0 \sec\theta & \text{if } \omega = \omega_0 \\ 0 & \text{if } \omega = \frac{1}{2}\omega_0 \\ \frac{1}{2}\sin 2\phi_0/(\cos^2\phi_0 \cos\theta - \mu) & \text{if } \omega \neq \omega_0, \omega \neq \frac{1}{2}\omega_0 \end{cases}$$

where $\mu = (\omega_0^2 - \omega^2)/(\omega_0^2 - 4\omega^2)$ and θ is the angle of scattering. To an observer facing the incident light, ϕ_0 is the minimum angle that the electric vector makes with the plane of scattering; similarly ϕ is the minimum angle that the electric vector of the scattered light viewed by an observer facing it makes with that plane. The consequences are discussed.

1. Introduction

The invention of lasers as sources of highly intense monochromatic electromagnetic waves and the advent of optoelectronic techniques for detecting radiation of very low intensity have led to great interest in the optical non-linear effects (Minck *et al* 1966) observed in the interaction of light with matter. Franken *et al* (1961) have generated second harmonics by a ruby laser beam in a crystal of quartz. The theoretical study of this phenomenon has been made for the case of free electrons (Vachaspati 1962, 1963, Brown and Kibble 1964, Prakash and Vachaspati 1967, Prakash 1969) as well as for bound electrons (Neugebauer 1959, 1963, Bali and Dutt 1965, Dutt *et al* 1966, Goyal and Prakash 1971, Verma and Vachaspati 1986). Similar studies of the problem (Brown and Kibble 1964, Englert and Rinehart 1983, Jafarpour 1975) using different techniques have confirmed the findings of Vachaspati (1962, 1963) and Prakash and Vachaspati (1967).

Recently, Englert and Rinehart (1983) have detected the second harmonic generation and studied its intensity. Using a polariser in front of the detector apparatus, the polarisation properties of second harmonic radiation by free electrons (Dhar Dwivedi and Vachaspati 1980) should be easy to verify experimentally. As the second harmonic generation (SHG) is enhanced by using bound electrons at resonances, the experimental detection of SHG in this case should be easier than that in the earlier case. The above experiment induces us to study the polarisation properties of second harmonics (2ω) in the scattering of plane polarised light by electrons bound by a simple harmonic force. The dependence of the scattering cross section on the angle of initial polarisation, ϕ_0, on scattering angle, θ, and on a, the ratio of the natural frequency of the bound electron to the frequency of the incident light, has also been discussed.

2. Equation of electron motion

The equation of motion

$$\ddot{z} - \gamma \dddot{z} + \omega_0^2 z = (e/m)[\boldsymbol{E}(z, t) + (\dot{z} \times \boldsymbol{H}(z, t))] \tag{1a}$$

of a harmonically bound electron of mass, m, and charge, e, in an external electromagnetic plane polarised monochromatic field of frequency ω with the wavevector $\boldsymbol{k} = \omega \boldsymbol{n}_0$,

$$\boldsymbol{E}(z, t) = \boldsymbol{E}_0 \cos(\omega t - \boldsymbol{k} \cdot z) \qquad \boldsymbol{H}(z, t) = \boldsymbol{H}_0 \cos(\omega t - \boldsymbol{k} \cdot z)$$

$$|\boldsymbol{E}_0| = |\boldsymbol{H}_0| = E_0 \tag{1b}$$

$$\boldsymbol{e}_0 = \boldsymbol{E}_0 / E_0 \qquad \boldsymbol{n}_0 = \boldsymbol{k}/\omega = (0, 0, 1) \qquad [\boldsymbol{n}_0, \boldsymbol{e}_0] = \boldsymbol{H}_0 / H_0 \qquad \gamma = \tfrac{2}{3} e^2/m$$

where γ is the damping constant, has been solved by using perturbation technique as a series in increasing powers of the amplitude of the incident fields:

$$z = \sum_{n=1}^{\infty} z^{(n)} \tag{2}$$

where $z^{(n)}$ denotes terms involving E_0^n.

On using relation $(1b)$, the equation of motion $(1a)$ reduces to the following.

(i) First-order equation

$$\ddot{z}^{(1)} - \gamma \dddot{z}^{(1)} + \omega_0^2 z^{(1)} = (e/m) E_0 \boldsymbol{e}_0 \cos \omega t. \tag{2a}$$

(ii) Second-order equation

$$\ddot{z}^{(2)} - \gamma \dddot{z}^{(2)} + \omega_0^2 z^{(2)} = \frac{1}{2} \frac{e^2 E_0^2}{\omega m^2} b_1 \boldsymbol{n}_0 \sin \phi_1 - \frac{1}{2} \frac{e^2 E_0^2}{\omega m^2} b_1 \boldsymbol{n}_0 \sin(2\omega t - \phi_1). \tag{2b}$$

(iii) Third-order equation

$$\begin{aligned} \ddot{z}^{(3)} - \gamma \dddot{z}^{(3)} + \omega_0^2 z^{(3)} = & \frac{1}{4} \frac{e^3 E_0^3}{\omega_0^2 m^3} b_1 \boldsymbol{e}_0 \sin \phi_1 \sin \omega t \\ & + \frac{3}{4} \frac{e^3 E_0^3}{\omega^2 m^3} b_1 b_2 \boldsymbol{e}_0 \cos(3\omega t - \phi_1 - \phi_2) \\ & + \frac{1}{4} \frac{e^3 E_0^3}{\omega^2 m^3} b_1 b_2 \boldsymbol{e}_0 \cos(\omega t - \phi_1 - \phi_2) \end{aligned} \tag{2c}$$

where $b_n = [(a^2 - n^2)^2 + \Gamma^2 n^6]^{-1/2}$ with $\Gamma = \gamma\omega$ and $a = \omega_0/\omega$.

3. Solution of the equation of motion

The explicit solutions of equations $(2a, b, c)$ in the strength of the external field are (Bali and Dutt 1965)

$$\omega z^{(1)} = \frac{eE_0}{\omega m} b_1 \boldsymbol{e}_0 \cos(\omega t - \phi_1) \tag{3a}$$

$$\omega z^{(2)} = \frac{1}{2} \frac{e^2 E_0^2}{\omega_0^2 m^2} b_1 \boldsymbol{n}_0 [\sin \phi_1 - b_2 a^2 \sin(2\omega t - \phi_1 - \phi_2)] \tag{3b}$$

and

$$\omega z^{(3)} = \frac{1}{4}\frac{e^3 E_0^3}{m^3 \omega_0^2 \omega} b_1 e_0 \{2b_1 \sin\phi_1 \sin(\omega t - \phi_1) + b_2 a^2 [b_1 \cos(\omega t - 2\phi_1 - \phi_2) + 3b_3 \cos(3\omega t - \phi_1 - \phi_2 - \phi_3)]\} \tag{3c}$$

where

$$\sin\phi_n = n^3 b_n \Gamma \qquad \cos\phi_n = (a^2 - n^2) b_n \qquad n = 1, 2, 3.$$

It is obvious from the above solutions that the electron oscillates not only with the frequency ω of the incident light but also with the frequencies that are integral multiples of it. The phase shifts, ϕ_n, are appreciable only for those frequencies which are integral submultiples of the natural frequency ω_0, i.e. for which $\omega = \omega_0/n$; in this case they become $\pm\pi/2$.

4. The radiation field

The electric and magnetic radiation fields of an oscillating electron in the direction of unit vector $\boldsymbol{n}$ from its mean position can be calculated with the help of the expressions ($3a, b, c$) at retarded time; they are (Vachaspati 1962)

$$\boldsymbol{E}^{\text{scatt}} = \boldsymbol{H}^{\text{scatt}} \times \boldsymbol{n} \qquad \boldsymbol{H}^{\text{scatt}} = \frac{e}{r}(\boldsymbol{n} \times \boldsymbol{M}) \qquad \boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}| \tag{4a}$$

$$\boldsymbol{M} = (1 - \boldsymbol{n}\cdot\boldsymbol{u})^{-2}(1 - u^2)^{-1}\{\boldsymbol{u}(1 - \boldsymbol{n}\cdot\boldsymbol{u})^{-1}[-(\dot{\boldsymbol{u}}\cdot\boldsymbol{n}) + (\boldsymbol{u}\cdot\dot{\boldsymbol{u}}) - (\boldsymbol{n}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\dot{\boldsymbol{u}}) + u^2(\boldsymbol{n}\cdot\dot{\boldsymbol{u}})][\dot{\boldsymbol{u}} - u^2\dot{\boldsymbol{u}} + (\boldsymbol{u}\cdot\dot{\boldsymbol{u}})\boldsymbol{u}]\} \tag{4b}$$

evaluated at the retarded time

$$t = x_0 - |\boldsymbol{x}| + \boldsymbol{n}\cdot\boldsymbol{z}(t)$$

where $\boldsymbol{u} = \mathrm{d}\boldsymbol{z}/\mathrm{d}t$ is the velocity of the electron and $\dot{\boldsymbol{u}} = \mathrm{d}^2\boldsymbol{z}/\mathrm{d}t^2$ is the acceleration of the electron. On using solutions ($3a, b, c$) equation ($4b$) reduces to the expression

$$\boldsymbol{M} = \boldsymbol{M}_\omega + \boldsymbol{M}_{2\omega} + \boldsymbol{M}_{3\omega} + \ldots \tag{4c}$$

where

$$\boldsymbol{M}_\omega = \frac{eE_0}{m} b_1 \boldsymbol{e}_0 \cos(\psi - \phi_1) - \frac{1}{8}\frac{e^3 E_0^3}{\omega^2 m^3} b_1^2 \{b_1 \boldsymbol{e}_0 \cos^2\alpha \cos(\psi - \phi_1) - 2b_2[(\boldsymbol{e}_0 \cos\theta - 2\boldsymbol{n}_0 \cos\alpha)\cos(\psi - \phi_2) + \boldsymbol{e}_0 \cos(\psi - 2\phi_1 - \phi_2)] - 4(1/a^2)\boldsymbol{e}_0(1 - \cos\theta)\sin\phi_1 \sin(\psi - \phi_1)\} \tag{4d}$$

$$\boldsymbol{M}_{2\omega} = -2\frac{e^2 E_0^2}{\omega m^2} b_1 [b_1 \boldsymbol{e}_0 \cos\alpha \sin 2(\psi - \phi_1) + b_2 \boldsymbol{n}_0 \sin(2\psi - \phi_1 - \phi_2)] \tag{4e}$$

$$\boldsymbol{M}_{3\omega} = \frac{9}{8}\frac{e^3 E_0^3}{\omega^2 m^3} b_1 \{6b_2 b_3 \boldsymbol{e}_0 \cos(3\psi - \phi_1 - \phi_2 - \phi_3) - b_1[3b_1 \boldsymbol{e}_0 \cos^2\alpha \cos 3(\psi - \phi_1) + 2b_2(\boldsymbol{e}_0 \cos\theta + 2\boldsymbol{n}_0 \cos\alpha)\cos(3\psi - \phi_1 - \phi_2 - \phi_3)]\}$$

$$\psi = \omega(x_0 - |\boldsymbol{x}|) \qquad \cos\alpha = (\boldsymbol{n}\cdot\boldsymbol{e}_0) = -\sin\theta\cos\phi_0 \qquad \cos\theta = \boldsymbol{n}\cdot\boldsymbol{n}_0. \tag{4f}$$

5. Polarisation properties of scattered 2ω radiation

5.1. Notation

To study the polarisation of the scattered 2ω radiation, we follow Dhar Dwivedi and Vachaspati (1980). We take the direction of the electric field as the direction of polarisation. Consider a plane (the plane of the scattering) containing the incident light and the observed light with unit vectors $\boldsymbol{n}_0$ and $\boldsymbol{n}$, respectively. The electric field, $\boldsymbol{E}^{\text{scatt}}$, of the scattered wave orthogonal to $\boldsymbol{n}$ is decomposed along unit vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ which are such that $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{n})$ form a right-handed coordinate system. If the second harmonic (2ω) scattered light is plane polarised, it can be written in the form

$$\boldsymbol{E}^{\text{scatt}} = E_0^{\text{scatt}}(\cos\phi\boldsymbol{\alpha}_1 + \sin\phi\boldsymbol{\alpha}_2)\cos(2\psi+\xi) \tag{5}$$

where

$$\boldsymbol{\alpha}_1 = \operatorname{cosec}\theta(\boldsymbol{n}_0 - \cos\theta\boldsymbol{n})$$

$$\boldsymbol{\alpha}_2 = \operatorname{cosec}\theta(\boldsymbol{n}\times\boldsymbol{n}_0) \qquad \cos\theta = \boldsymbol{n}_0\cdot\boldsymbol{n}$$

and ξ is the phase angle. ϕ is the angle of polarisation of the light scattered through scattering angle θ which the electric vector makes with $\boldsymbol{\alpha}_1$. We do not distinguish between the angles ϕ and $\pi+\phi$ and ϕ varies from 0 to π.

The incident electric vector, $\boldsymbol{E}^{\text{inc}}$, can similarly be decomposed along unit vectors $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ which, along with $\boldsymbol{n}_0$, form a right-handed coordinate system. One can write equation (1b) as

$$\boldsymbol{E}^{\text{inc}} = E_0^{\text{inc}}(\cos\phi_0\boldsymbol{\beta}_1 + \sin\phi_0\boldsymbol{\beta}_2)\cos(\omega t - \boldsymbol{k}\cdot z) \tag{6}$$

where $\boldsymbol{\beta}_1 = -\operatorname{cosec}\theta(\boldsymbol{n} - \cos\theta\boldsymbol{n}_0)$, $\boldsymbol{\beta}_2 = \boldsymbol{\alpha}_2$ and ϕ_0 is the angle of polarisation of the incident light which the electric vector makes with $\boldsymbol{\beta}_1$ (see figure 1).

5.2. Electric field for second harmonic (2ω)

On using equation (4e), equation (4a) then reduces to the expression

$$\boldsymbol{E}_{2\omega}^{\text{scatt}} = -\frac{2e^3E_0^2}{r\omega m^2}b_1[b_1(\cos\alpha\boldsymbol{e}_0 - \cos^2\alpha\boldsymbol{n})\sin 2(\psi-\phi_1) + b_2(\boldsymbol{n}_0 - \cos\theta\boldsymbol{n})\sin(2\psi-\phi_1-\phi_2)]. \tag{7}$$

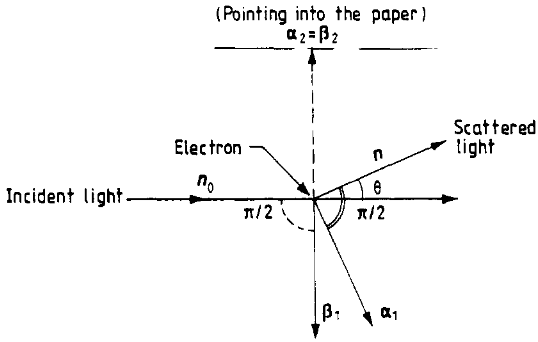

Figure 1. The electric vector of incident light is resolved along mutually perpendicular unit vectors ($\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ and $\boldsymbol{n}_0$) and that of the scattered 2ω light is resolved along the unit vectors ($\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ and $\boldsymbol{n}$). Unit vectors $\boldsymbol{\beta}_1, \boldsymbol{\alpha}_1, \boldsymbol{n}_0$ and $\boldsymbol{n}$ are in the plane of the scattering and $\boldsymbol{\beta}_2 = \boldsymbol{\alpha}_2$ is perpendicular to the plane of scattering pointing into the paper.

To study the polarisation properties of the second harmonic scattered light, we resolve $\boldsymbol{e}_0$ and $\boldsymbol{n}_0$ (direction of propagation of the incident beam) along the three axes $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and $\boldsymbol{n}$. Thus we obtain

$$\boldsymbol{e}_0 = \cos\theta\cos\phi_0\boldsymbol{\alpha}_1 + \sin\phi_0\boldsymbol{\alpha}_2 - \sin\theta\cos\phi_0\boldsymbol{n} \tag{8a}$$

and

$$\boldsymbol{n}_0 = \sin\theta\boldsymbol{\alpha}_1 + \cos\theta\boldsymbol{n}. \tag{8b}$$

Substituting equations (8*a*, *b*) in equation (7), we have

$$E_{2\omega}^{\text{scatt}} = A\boldsymbol{S} \qquad A = \frac{2}{r}\left(\frac{e^3E_0^2}{\omega m^2}\right)b_1\sin\theta$$

$$\boldsymbol{S} = [b_1\cos\phi_0(\cos\theta\cos\phi_0\boldsymbol{\alpha}_1 + \sin\phi_0\boldsymbol{\alpha}_2)\sin 2(\psi - \phi_1) - b_2\boldsymbol{\alpha}_1\sin(2\psi - \phi_1 - \phi_2)]. \tag{9}$$

It is to be noticed that $\boldsymbol{S}$, to which $\boldsymbol{E}^{\text{scatt}}$ is proportional through a factor A, is not of the simple form (5) given in § 5, because in this case the light is, in general, elliptically polarised while the expression (5) is for plane polarised light. That the ellipticity is very small and is, in fact, proportional to the very small factor Γ, is a conclusion at which we arrive only after a fairly long calculation. One can write equation (9) as

$$\boldsymbol{S} = \boldsymbol{S}_c\cos(2\psi - 2\phi_1 + \xi) + \boldsymbol{S}_s\sin(2\psi - \phi_1 - \phi_2 + \xi) \tag{10}$$

with $\boldsymbol{S}_c \cdot \boldsymbol{S}_s = 0$ (the orthogonality condition) where

$$\boldsymbol{S}_c = -(B_1\boldsymbol{\alpha}_1 + B_2\boldsymbol{\alpha}_2)\sin\xi - B_3\cos\xi\boldsymbol{\alpha}_1$$

and (11)

$$\boldsymbol{S}_s = (B_1\boldsymbol{\alpha}_1 + B_2\boldsymbol{\alpha}_2)\cos\xi - B_3\sin\xi\boldsymbol{\alpha}_1.$$

In equations (10) and (11), ξ is essentially the ellipticity parameter. From the above expressions, we have the following relation:

$$\cot 2\xi = -(B_1^2 + B_2^2 - B_3^2)/B_1B_3 \tag{12}$$

$$\begin{aligned} B_1 &= b_1\cos^2\phi_0\cos\theta - b_2\cos(\phi_1 - \phi_2) \\ B_2 &= b_1\sin\phi_0\cos\phi_0 \\ B_3 &= b_2\sin(\phi_1 - \phi_2) \end{aligned} \tag{13}$$

where

$$\cos(\phi_1 - \phi_2) = b_1b_2[(a^2-1)(a^2-4) + 8\Gamma^2]$$

$$\sin(\phi_1 - \phi_2) = b_1b_2(4 - 7a^2)\Gamma.$$

Let us discuss some special cases of interest.

5.2.1. For resonance at $a = 1$, i.e. $\omega = \omega_0$. After a straightforward but tedious calculation, we find that the ellipticity parameter ξ is given by the expression

$$\cos\xi = \frac{\Gamma}{3}\,\frac{16\cos^2\phi_0\cos\theta(\sin^2\phi_0 + \cos^2\phi_0\cos^2\theta) + \sin^2\phi_0}{\cos\phi_0(\sin^2\phi_0 + \cos^2\phi_0\cos^2\theta)}. \tag{14}$$

For very small Γ, $\cos\xi = 0$, i.e. $\xi = \pi/2$, except for some special cases. The expression for double frequency electric field can be written as

$$S = \frac{1}{\Gamma}\cos\phi_0[\cos\phi_0\cos\theta\boldsymbol{\alpha}_1 + \sin\phi_0\boldsymbol{\alpha}_2]\sin f$$
$$f = 2\psi - 2\phi_1 \qquad \tan\phi = \tan\phi_0\sec\theta. \tag{15}$$

The second harmonic light is thus plane polarised and the angle of polarisation ϕ depends both on the angle of scattering θ and on the angle of initial polarisation ϕ_0 and has been plotted in figure 2 for different θ and ϕ_0. In particular, it is not polarised in the plane of scattering except in the case when the coefficient of $\boldsymbol{\alpha}_2$ vanishes, i.e. when $\phi_0 = 0$. The scattered light is polarised in the perpendicular plane either when the incident polarisation angle ϕ_0 is equal to $\pi/2$ irrespective of the value of θ or when the scattering angle θ is equal to $\pi/2$ irrespective of the value of ϕ_0.

The cross section is plotted in terms of ϕ_0 and θ in figure 3. The expression for the cross section is given by

$$\left(\frac{\mathrm{d}\sigma^{(2\omega)}}{\mathrm{d}\Omega}\right)_{\text{pol light}} = \left(\frac{4}{\Gamma^4}\right)\frac{e^6E_0^2}{m^4\omega^2}[\tfrac{1}{4} - (\sin^2\theta\cos^2\phi_0 - \tfrac{1}{2})^2]. \tag{16}$$

We see from (16) that the scattering is symmetric about $\theta = \pi/2$ for all ϕ_0 as is obvious from figure 3. The cross section vanishes at $\theta = 0$, and $\theta = \pi$ for all ϕ_0.

5.2.2. For resonance at $\omega = \frac{1}{2}\omega_0$. The ellipticity in this case is as

$$\sin\xi = \tfrac{4}{3}\sin 2\phi_0\Gamma$$
$$\cos\xi = 1 - \tfrac{8}{9}\sin^2 2\phi_0\Gamma^2. \tag{17}$$

We see that the ellipticity is directly proportional to a very small factor Γ which can be neglected. We have $\sin\xi \simeq 0$, i.e. $\xi = 0$ and

$$S = \frac{1}{8\Gamma}\boldsymbol{\alpha}_1\cos f. \tag{18}$$

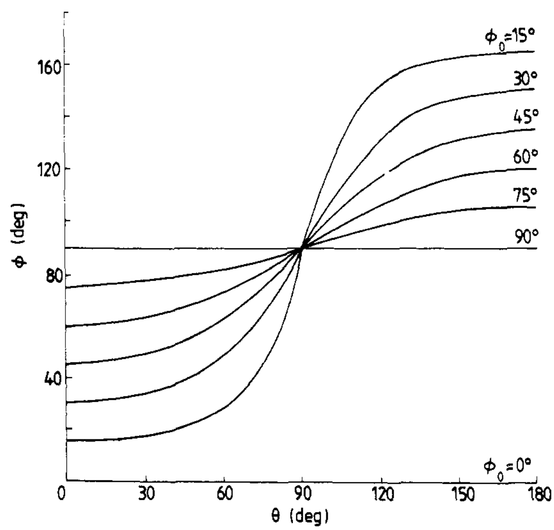

Figure 2. Variation of ϕ as a function of θ for different ϕ_0 at resonance $\omega = \omega_0$, $\tan\phi = \tan\phi_0\sec\theta$.

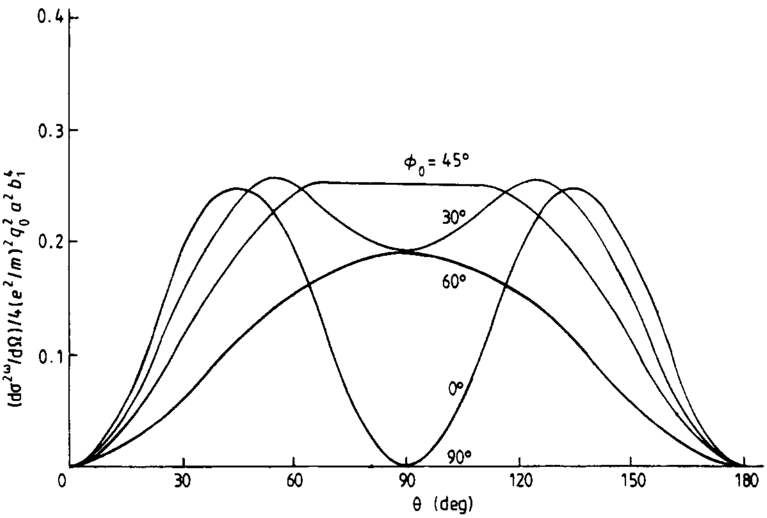

Figure 3. Variation of the cross section as a function of θ for various ϕ_0 at $\omega = \omega_0$, i.e. $a = 1$ and $q_0 = eE_0/m\omega_0$. It vanishes at $\theta \simeq 90°$ for $\phi_0 = 0°$ and attains maximum value at $\theta \simeq 35°$ for $\phi_0 = 30°$.

The scattered 2ω radiation is linearly polarised and its electric field vector is perpendicular to $\boldsymbol{n}$ (the unit vector along the propagation of scattered light) in the plane of scattering. From equation (18), we have $\tan\phi = 0$, i.e. $\phi = 0$. This means that the second harmonic light is always polarised in the plane of the scattering, no matter what the scattering angle and the incident polarisation are. This is the best method for obtaining plane polarised scattered 2ω light. The scattering cross section at $\omega = \frac{1}{2}\omega_0$ is given by

$$\left(\frac{d\sigma^{(2\omega)}}{d\Omega}\right)_{\text{pol light}} = \frac{1}{144\Gamma^2}\frac{e^6 E_0^2}{\omega^2 m^4}\sin^2\theta. \tag{19}$$

It is seen that the cross section is independent of the initial polarisation ϕ_0 and only depends on the scattering angle θ. The behaviour of the cross section for various values of θ is illustrated in figure 4. The scattering is symmetric about $\theta = \pi/2$. The cross section is a maximum at $\theta = \pi/2$ and vanishes at $\theta = 0$ and $\theta = \pi$. The cross section for resonance at $\omega = \omega_0$ is much greater than in the case of resonance at $\omega = \frac{1}{2}\omega_0$ because of $1/\Gamma^4$ dependence.

5.2.3. Far from resonances. If we are far from the resonances, i.e. $|a-1| \gg \Gamma$, $|a-2| \gg \Gamma$, then $b_1 \simeq \omega^2/(\omega_0^2 - \omega^2)$ and the phase shift becomes zero. The electric field for the second harmonic in this case becomes

$$\boldsymbol{S} = b_1\{[\cos^2\phi_0\cos\theta - (a^2-1)/(a^2-4)]\boldsymbol{\alpha}_1 + \sin\phi_0\cos\phi_0\boldsymbol{\alpha}_2\}\sin 2\psi. \tag{20}$$

The intensity of scattered light is very small in this case of non-resonance. We see from expression (20) that the light is plane polarised and its angle of polarisation ϕ is given by

$$\tan\phi = \frac{\sin\phi_0\cos\phi_0}{\cos\theta\cos^2\phi_0 - (a^2-1)/(a^2-4)} \qquad 0 \leq \phi < \pi. \tag{21}$$

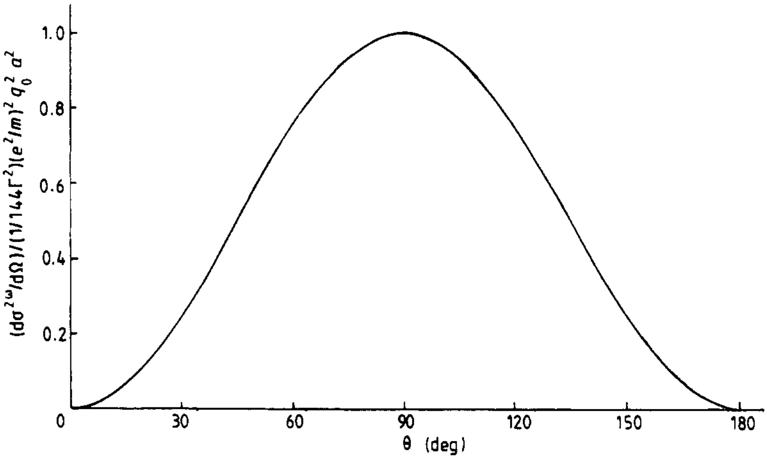

Figure 4. Angular variation of the scattering cross section for second harmonic at $\omega = \frac{1}{2}\omega_0$, i.e. $a = 2$ and $q_0 = eE_0/m\omega_0$. It is observed that the cross section is a maximum at $\theta = 90°$.

The nature of the final polarisation for various typical values of a, θ and ϕ_0 is illustrated in figures 5 and 6. We see that if $\phi_0 = 0$ or $\pi/2$ then $\phi = 0$. This means that, if the incident light is polarised in the plane of scattering or perpendicular to it, then the 2ω scattered light is polarised in the plane of scattering, no matter what the angle of scattering is.

The scattered light would be polarised perpendicularly to the plane of the scattering if

$$\cos^2\phi_0 \cos\theta = (a^2 - 1)/(a^2 - 4). \tag{22}$$

This requires $(a^2-1)/(a^2-4)$ to lie between -1 and 1, which means that $a \leqslant \sqrt{\frac{5}{2}}$ and,

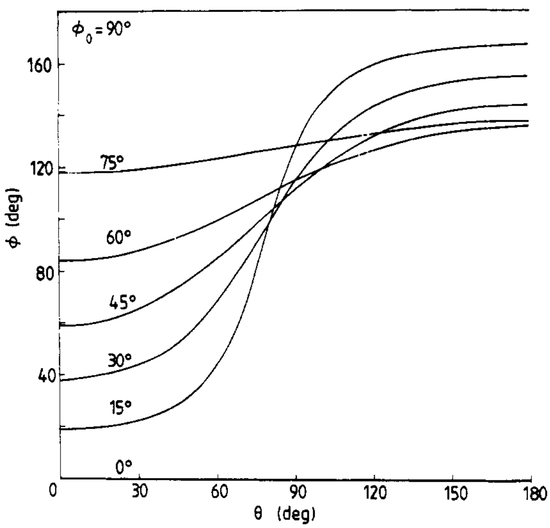

Figure 5. Polarisation of the scattered 2ω light, ϕ, plotted against the angle of scattering θ for various incident polarisation ϕ_0 at $a = 0.5$ ($\omega \neq \omega_0$, $\omega \neq \frac{1}{2}\omega_0$).

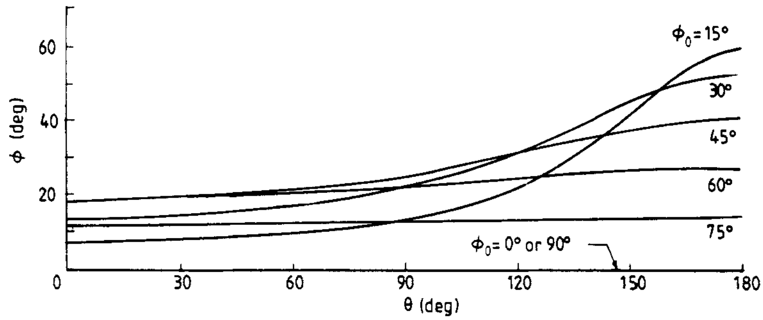

Figure 6. Polarisation of the scattered 2ω light, ϕ, plotted against the angle of scattering θ for various incident polarisation ϕ_0 at $a = 1.6$ ($\omega \neq \omega_0$, $\omega \neq \frac{1}{2}\omega_0$).

since $a = \omega_0/\omega$, this can also be written

$$\omega \geqslant 0.63\omega_0. \tag{23}$$

It is seen from the above relation that, when the frequency of incident light is equal to or greater than 0.63 times the natural frequency of bound electrons, the scattered light can be polarised perpendicularly to the plane of the scattering for particular values of ϕ_0.

The expression for the cross section at $|\omega - \omega_0| \gg \Gamma$, $|\omega - \frac{1}{2}\omega_0| \gg \Gamma$ in terms of ϕ_0, θ and a, is written as

$$\left(\frac{\mathrm{d}\sigma^{(2\omega)}}{\mathrm{d}\Omega}\right) = \frac{4e^6 E_0^2 \sin^2\theta}{\omega^2 m^4 (a^2-1)^4}[(\mu - \cos\theta\cos^2\phi_0)^2 + \tfrac{1}{4}\sin^2 2\phi_0] \tag{24}$$

and has been plotted in figure 7 for different values of ϕ_0, θ and μ. For $|\mu| < 1$ then $a < \sqrt{\frac{5}{2}}$ (see figure 8). μ may be neglected and the angular dependence of the cross

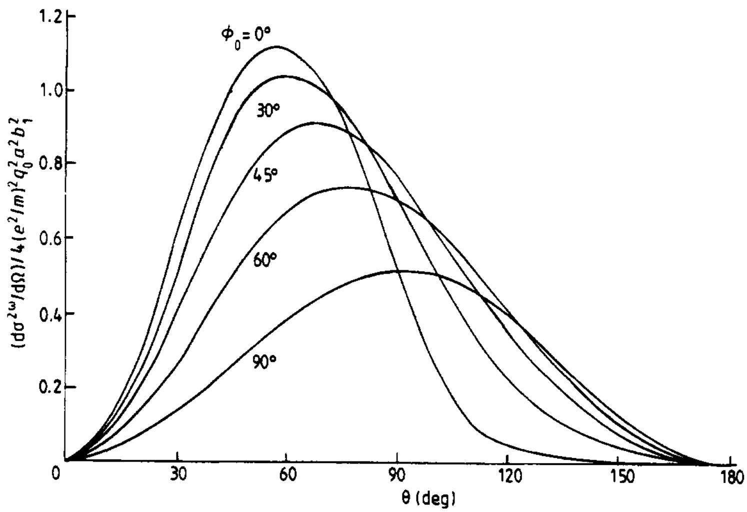

Figure 7. The differential scattering cross section for second harmonic against θ and ϕ_0 for $a = 1.5$ ($\omega \neq \omega_0$, $\omega \neq \frac{1}{2}\omega_0$, $q_0 = eE_0/m\omega_0$). It is observed that the cross section attains its maximum value at $\theta \simeq 57°$, $\phi_0 = 0°$.

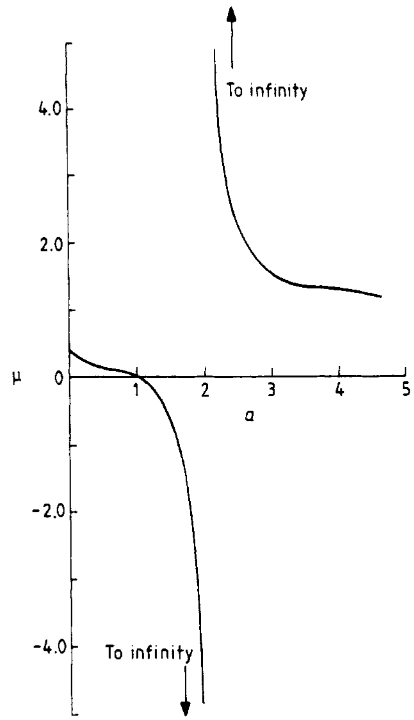

Figure 8. Variation of μ with $a = \omega_0/\omega$, the ratio of the frequency of bound electron to the frequency of the incident beam.

section is essentially given by

$$\sin^2\theta\cos^2\phi_0(\cos^2\theta\cos^2\phi_0+\sin^2\phi_0).$$

If $|\mu|$ exceeds 1, and is rather large as happens for $a>\sqrt{\frac{5}{2}}$, then the cross section is essentially proportional to $\sin^2\theta\mu^2$ and gets more or less independent of ϕ_0. The term $-2\mu\cos\theta\cos^2\phi_0\sin^2\theta$ adds a ripple to the background value $\mu^2\sin^2\theta$ and another smaller ripple arises from the rest of the term independent of μ. For intermediate values of $|\mu|$, when it is neither too small to be neglected nor too large to be the dominant term, the behaviour of the cross section is more complicated. This happens when a is in the neighbourhood of $\sqrt{\frac{5}{2}}\simeq 1.58$ or when a exceeds, say, 4 (so that $\mu\to 1$).

6. Concluding remarks

As shown above the second harmonic beam into which the plane polarised electromagnetic radiation is scattered is also nearly plane polarised. In the case of non-resonance at $|\omega-\omega_0|\gg 0$ and $|\omega-\frac{1}{2}\omega_0|\gg 0$, the scattered 2ω radiation is polarised in the plane of the scattering when the incident light is polarised in the plane of scattering or perpendicular to it, no matter what the angle of scattering is. The scattered light can have perpendicular polarisation only in some special cases which are discussed. In the case of resonance at $\omega=\frac{1}{2}\omega_0$, the second harmonic is always polarised in the plane of scattering, no matter what the polarisation of the incident light or the angle of scattering are. The polarisation angle ϕ is given by equation (15) in the case of resonance at $\omega=\omega_0$.

Acknowledgments

We are grateful to Professor Bipin K Agarwal for providing facilities in the department and also to Drs H Prakash, C M Bhandari and V K Agarwal for their helpful suggestions. One of the authors (VPV) is grateful to CSIR for the award of a Postdoctoral Research Associateship.

References

Bali L M and Dutt J 1965 *Nuovo Cimento* **35** 805
Brown L S and Kibble T W B 1964 *Phys. Rev.* **133** A705
Dhar Dwivedi A K and Vachaspati 1980 *Ind. J. Phys.* B **54** 180
Dutt J, Bali L M and Vachaspati 1966 *Ind. J. Pure Appl. Phys.* **4** 137
Englert T J and Rinehart F A 1983 *Phys. Rev.* A **28** 1539
Franken P A, Hill A E, Peters C W and Weinreich G 1961 *Phys. Rev. Lett.* **7** 118
Goyal R and Prakash H 1971 *Ind. J. Pure Appl. Phys.* **9** 75
Jafarpour M 1975 *PhD Thesis* University of Wyoming
Minck R W, Terheme R W and Wang C C 1966 *Appl. Opt.* **5** 1955
Neugebauer T 1959 *Z. Phys.* **155** 380
—— 1963 *Z. Phys.* **16** 217
Prakash H 1969 *Ind. J. Pure Appl. Phys.* **7** 77
Prakash H and Vachaspati 1967 *Ind. J. Pure Appl. Phys.* **5** 21
Verma V P and Vachaspati 1986 *Physica* **144C** 227
Vachaspati 1962 *Phys. Rev.* **128** 664
—— 1963 *Phys. Rev.* **130** 2598